Journal of Statistical Physics, Vol. 111, Nos. 1/2, April 2003 (© 2003)

Proof of an Asymptotic Property of Self-Similar Solutions of the Boltzmann Equation for Granular Materials

A. V. Bobylev,[1] C. Cercignani,[2] and G. Toscani[3]

Received June 18, 2002

We consider a question related to the kinetic theory of granular materials. The model of hard spheres with inelastic collisions is replaced by a Maxwell model, characterized by a collision frequency independent of the relative speed of colliding particles. Our main result is that, in the space-homogeneous case, a self-similar asymptotics holds, as conjectured by Ernst–Brito. The proof holds for any initial distribution function with a finite moment of some order greater than two.

KEY WORDS: Granular material; Boltzmann equation; self-similar solutions.

1. INTRODUCTION

The concept of self-similar solutions of nonlinear evolution equations and their role in self-similar asymptotics appear to be very useful and fruitful in various domains of mathematical physics. Although it is not too difficult to describe (on the basis of the properties of the symmetry group of the equation) possible classes of self-similar solutions, the most difficult problem is to prove that some of these solutions play an asymptotic role for a wide class of initial conditions.

The very first results of this type in the kinetic theory of gases were recently obtained in refs. 1 and 2. The self-similar solutions (with infinite energy) of the Boltzmann equation for Maxwell molecules were constructed

[1] Division of Engineering Sciences, Physics and Mathematics, Karlstad University, Karlstad, Sweden.
[2] Dipartimento di Matematica, Politecnico di Milano, Milano, Italy; e-mail: carcer@mate.polimi.it
[3] Dipartimento di Matematica, Università di Pavia, Pavia, Italy.

0022-4715/03/0400-0403/0

(two of them in an explicit form) and it was proved that these solutions do describe a large time asymptotics for certain classes of initial conditions.

Independently, Ernst and Brito[3, 4] (see also refs. 5 and 6) studied the large time behavior of the solutions of the so-called inelastic Maxwell models[7] and conjectured that the solutions must have a self-similar asymptotics. The conjecture seemed to be quite non-trivial since the corresponding self-similar solution (whose possible existence was just indicated in refs. 3 and 4) would have a power-like high energy tail. It was already known that, for almost all values of the restitution coefficient, self-similar solutions with finite moments of all orders cannot be positive.[7]

The first proof of Ernst–Brito conjecture for a subclass of isotropic initial conditions which includes Maxwellians was given in ref. 8. It was clear, however, that the restrictions on initial conditions used in ref. 8 are related to mathematical technicalities, not to physics. The main goal of the present paper is to prove the conjecture for a very general class of initial conditions.

The paper is organized as follows. We formulate the general problem and related known results (Propositions 2.1 and 2.2) in Section 2. Then, in Section 3, we perform some necessary calculations of moments. Section 4 is devoted to self-similar solutions (Proposition 4.3) and an "incomplete" proof of the conjecture (lemma at the end of Section 4). The proof is completed in Section 5 which ends with a theorem which contains our main results.

One can summarize our main result by saying that we prove that the self-similar asymptotics (Ernst–Brito conjecture) holds for any initial distribution function having "something more" than finite energy (namely, a finite moment of some order greater than two). It seems plausible that this restriction, typical also for the classic (elastic) Boltzmann equation, cannot be weakened.[9]

We stress that all our results relate to the spatially homogeneous systems without external forces. The questions on how to use them for driven and/or spatially inhomogeneous systems remain open. We hope to return to these questions in the future.

2. STATEMENT OF THE PROBLEM

The d-dimensional ($d = 2, 3, \ldots$) Maxwell model for inelastic particles (in the space homogeneus case) is defined as follows.[8] The one-particle distribution function $f(\mathbf{v}, t)$ ($\mathbf{v} \in \mathbb{R}^d$, $t \in \mathbb{R}_+$) satisfies the Boltzmann-type equation

$$\frac{\partial f}{\partial t} = Q(f, f); \qquad f_{|t=0} = f_0(t), \tag{2.1}$$

such that the weak form reads

$$\frac{\partial}{\partial t}(f, h) = \frac{1}{2} \int_{\mathbb{R}^d} \int_{\mathbb{R}^d} \int_{S^{d-1}} f(t, \mathbf{v})\, f(t, \mathbf{w})\, B\left(\frac{(\mathbf{u} \cdot \mathbf{n})}{|\mathbf{u}|}\right)$$
$$\times [h(\mathbf{v}') + h(\mathbf{w}') - h(\mathbf{v}) - h(\mathbf{w}))]\, d\mathbf{n}\, d\mathbf{w}\, d\mathbf{v}. \tag{2.2}$$

Here $h(\mathbf{v}) \in C_\infty(\mathbb{R}^d)$ is a test function, $B(\cos\theta) \geqslant 0$ the collision kernel normalized by

$$\int_{S^{d-1}} B(\boldsymbol{\omega} \cdot \mathbf{n})\, d\mathbf{n} = 1. \tag{2.3}$$

Other notations are given by

$$(f, h) = \int_{\mathbb{R}^d} f(\mathbf{v})\, h(\mathbf{v}),$$
$$\mathbf{v}' = \mathbf{v} - \frac{1+e}{2}(\mathbf{u} \cdot \mathbf{n})\, \mathbf{n}; \quad \mathbf{w}' = \mathbf{w} + \frac{1+e}{2}(\mathbf{u} \cdot \mathbf{n})\, \mathbf{n} \qquad \mathbf{u} = \mathbf{v} - \mathbf{w}. \tag{2.4}$$

where $0 < e < 1$ is the restitution coefficient by which the after-collision velocities $\mathbf{v}'$, $\mathbf{w}'$ can be computed.

A connection between the Maxwell model (2.1) with the usual kinetic equation for d-dimensional inelastic hard spheres is explained in detail in ref. 8 (see also ref. 7 for the case $d = 3$). The corresponding collision kernel (related to the hard sphere model) reads

$$B(\cos\theta) = A_d\, |\cos\theta|, \tag{2.5}$$

where the constant factor A_d is such that the condition (2.3) is satisfied. For the sake of generality we consider an arbitrary kernel $B \in L_+([-1, 1])$ satisfying Eq. (2.3).

The main advantage of the inelastic Maxwell model (2.1), (2.2) is the well known simplification[7,8] when we use the Fourier transform. If we denote

$$\phi(\mathbf{k}, t) = (f, e^{-i\mathbf{k} \cdot \mathbf{v}}), \quad \phi_0(\mathbf{k}) = (f_0, e^{-i\mathbf{k} \cdot \mathbf{v}}), \qquad \mathbf{k} \in \mathbb{R}^d. \tag{2.6}$$

Then the equation for $\phi(\mathbf{k}, t)$ reads as follows (see ref. 8 for details)

$$\frac{\partial \phi}{\partial t} = \hat{Q}(\phi, \phi) = \int_{S^{d-1}} d\mathbf{n}\, B(\hat{\mathbf{k}} \cdot \mathbf{n})[\phi(\mathbf{k}_+)\, \phi(\mathbf{k} - \mathbf{k}_+) - \phi(0)\, \phi(\mathbf{k})], \tag{2.7}$$

where

$$\mathbf{k}_+ = z(\mathbf{k}\cdot\mathbf{n})\,\mathbf{n}, \qquad z = \frac{1+e}{2}, \qquad \hat{\mathbf{k}} = \frac{\mathbf{k}}{|\mathbf{k}|} \qquad \phi_{|t=0} = \phi_0(\mathbf{k})$$

Without loss of generality we can assume that

$$(f_0, 1) = 1, \qquad (f_0, \mathbf{v}) = 0, \qquad (f_0, |\mathbf{v}|^2) = d, \tag{2.8}$$

and thus

$$\phi_0(0) = 1, \qquad (\nabla\phi_0)_{|\mathbf{k}=0} = 0, \qquad (\Delta\phi_0)_{|\mathbf{k}=0} = -d, \tag{2.9}$$

We remark that Eqs. (2.2), (2.4), (2.7) coincide with the usual (elastic) Boltzmann equation for pseudo-Maxwell molecules, if $z = e = 1$. This case was studied in detail by the Fourier transform since 1975.[1, 2, 8–11] All the results on uniqueness and existence of solutions to the Cauchy problem (2.1), (2.7) can be easily extended to the inelastic case, all proofs (based on Wild's sums, etc.) remaining practically the same. Therefore we omit proofs and just formulate the main results:

Proposition 2.1. Let $f_0(\mathbf{v})$ be a generalized density of the probability measure (i.e., a non-negative measure in $\mathbb{R}^d$ normalized to unity) satisfying Eqs. (2.8). Then there exists a unique characteristic function (the Fourier transform of a probability measure) $\phi(\mathbf{k}, t)$ which solves Problem (2.7). Moreover, $\phi(\mathbf{k}, t)$ is a unique solution of this problem in the class of locally bounded functions.

Proposition 2.2. If, in addition to the assumptions in Proposition 1, there exists $\delta > 0$ such that

$$(f_0, |\mathbf{v}|^{2(1+\delta)}) < \infty, \tag{2.10}$$

then $\phi(\mathbf{k}, t) \in C_2(\mathbb{R}^d)$ for all $t \geqslant 0$ and

(i) the following asymptotic formula holds:

$$\phi(\mathbf{k}, t) = 1 - \tfrac{1}{2}\, p_{\alpha\beta} k_\alpha k_\beta + O(|\mathbf{k}|^{2(1+\delta)}), \qquad |\mathbf{k}| \to 0 \tag{2.11}$$

(here and below a summation over repeated Cartesian indices $\alpha, \beta = 1,\ldots, d$ is assumed);

(ii) $|\phi(\mathbf{k}, t)| \leqslant 1$ for all $t \geqslant 0$ and $\phi(\mathbf{k}, t) \to_{t\to\infty} 1$ uniformly in $\mathbb{R}^d$.

Remark. The condition (2.10) is needed only for the asymptotic equality (2.11) which plays an important role in the self-similar asymptotics (see below).

We note that all properties of the (generalized) solution $f(\mathbf{v}, t)$ of the kinetic equation (2.1) can be easily obtained from the properties of the characteristic function $\phi(\mathbf{k}, t) = \mathscr{F}[f]$ on the basis of the general theory of characteristic functions (see, e.g., ref. 12). In particular, the pointwise convergence $\phi_n(\mathbf{k}) \to \phi(\mathbf{k})$ of the sequence $\{\phi_n(\mathbf{k})\}$ to the function $\phi(\mathbf{k})$, continuous at $\mathbf{k} = 0$ with $\phi(0) = 1$, implies the convergence of the corresponding generalized densities $f_n(\mathbf{v}) = \mathscr{F}^{-1}[\phi_n]$ in the sense of probability measures and the existence of the limiting measure. Thus, the property (ii) means that

$$f(\mathbf{v}, t) \xrightarrow[t \to \infty]{} \delta(\mathbf{v}) \tag{2.12}$$

in sense described above.

Thus in the sequel we shall mostly discuss properties of $\phi(\mathbf{k}, t)$, not of $f(\mathbf{v}, t)$.

On one hand, everything seems very simple and clear; any initial distribution function $f_0(\mathbf{v})$ satisfying Eqs. (2.8), (2.10) relaxes, as $t \to \infty$, to a delta function (2.12) (space homogeneous cooling). On the other hand, the solution $f(\mathbf{v}, t)$ is expected to have a very interesting asymptotics first conjectured by Ernst and Brito[3, 4]: if we denote

$$T(t) = \frac{1}{d}(f, |\mathbf{v}|^2), \qquad f(\mathbf{v}, t) = [T(t)]^{-d/2} F\left(\frac{\mathbf{v}}{\sqrt{T(t)}}, t\right), \tag{2.13}$$

then the re-scaled solution $F(\mathbf{w}, t)$ tends (in a certain sense), as $t \to \infty$, to a function $F_\infty(|\mathbf{w}|)$, which, in turn, is a steady solution of the re-scaled version of Eq. (2.1) (or, equivalently, Eq. (2.13) with $F_\infty(|\mathbf{w}|)$ instead of $F(\mathbf{w}, t)$ defines a self-similar solution of Eq. (2.1)).

The first proof of the conjecture (for an important sub-class of isotropic functions $f_0(|\mathbf{v}|)$ that includes Maxwellian distributions) was recently given in ref. 8. The main goal of the present paper is to give a complete proof of the Ernst–Brito conjecture for any initial data satisfying the assumption of Proposition 2.2. In other words, the conjecture (where the convergence $F(\mathbf{w}, t) \to_{t \to \infty} F_\infty(|\mathbf{w}|)$ is understood in the sense of probability measures) is true for any initial distribution function $f_0(\mathbf{v})$ with a finite moment of order greater than 2 (the assumption (2.10) is the only essential restriction we shall need).

3. EVOLUTION OF THE FIRST MOMENTS

We begin with an exact calculation of the tensor $p_{\alpha\beta}$ in Eq. (2.11) (see ref. 13 for the case $d=3$). By substituting the asymptotic expansion (2.11) into Eq. (2.7), we obtain for the terms of second order in $\mathbf{k}$:

$$k_\alpha k_\beta \frac{dp_{\alpha\beta}}{dt} = p_{\alpha\beta} \hat{L} k_\alpha k_\beta \tag{3.1}$$

$$\hat{L}\phi(\mathbf{k}, t) = \langle \phi[z(\mathbf{k}\cdot\mathbf{n})\,\mathbf{n}] + \phi[\mathbf{k} - z(\mathbf{k}\cdot\mathbf{n})\,\mathbf{n}] - \phi(\mathbf{k})\rangle, \tag{3.2}$$

in the notation

$$\langle \Phi(\mathbf{k}, \mathbf{n})\rangle = \int_{S^{d-1}} d\mathbf{n}\, B(\hat{\mathbf{k}}\cdot\mathbf{n})\, \Phi(\mathbf{k}, \mathbf{n}). \tag{3.3}$$

The operator $\hat{L}$ is obviously isotropic (invariant under rotations of $\mathbb{R}^d$) and invariant under scaling transformations $\mathbf{k} \to \alpha\mathbf{k}$, $\alpha = \text{const}$. These properties define the polynomial eigenfunctions of $\hat{L}$.

In particular,

$$\hat{L}\,|\mathbf{k}|^2 = -\lambda_0\,|\mathbf{k}|^2, \qquad \hat{L}\left(k_\alpha k_\beta - \frac{|\mathbf{k}|^2}{d}\,\delta_{\alpha\beta}\right) = -\lambda_1\left(k_\alpha k_\beta - \frac{|\mathbf{k}|^2}{d}\,\delta_{\alpha\beta}\right) \tag{3.4}$$

where

$$\begin{aligned} \lambda_0 &= 2z(1-z)\,\langle(\hat{\mathbf{k}}\cdot\mathbf{n})^2\rangle, \\ \lambda_1 &= \lambda_0 + \frac{2dz^2}{d-1}\,\langle(\hat{\mathbf{k}}\cdot\mathbf{n})^2[1-(\hat{\mathbf{k}}\cdot\mathbf{n})^2]\rangle. \end{aligned} \tag{3.5}$$

The tensor $p_{\alpha\beta}$ can be written as

$$p_{\alpha\beta} = (f, v_\alpha v_\beta) = \left(f, v_\alpha v_\beta - \frac{|\mathbf{v}|^2}{d}\,\delta_{\alpha\beta}\right) + T(t)\,\delta_{\alpha\beta} = q_{\alpha\beta} + T(t)\,\delta_{\alpha\beta}, \qquad q_{\alpha\alpha} = 0 \tag{3.6}$$

in the notation (2.13). Then Eq. (3.1), (3.4), (3.6) yield (note that $T(0)=1$ by the assumption (2.8))

$$T(t) = e^{-\lambda_0 t}, \qquad q_{\alpha\beta}(t) = q_{\alpha\beta}(0)\,e^{-\lambda_1 t}. \tag{3.7}$$

Thus we obtain the expansion (2.11) in a more explicit form:

$$\phi(\mathbf{k}, t) = 1 - \frac{|\mathbf{k}|^2}{2} e^{-\lambda_0 t} - \frac{1}{2} q_{\alpha\beta}(t) \left(k_\alpha k_\beta - \frac{|\mathbf{k}|^2}{d} \delta_{\alpha\beta} \right)$$
$$+ O(|\mathbf{k}|^{2(1+\delta)}) e^{-\lambda_1 t}, \qquad \delta > 0, \quad |\mathbf{k}| \to 0. \tag{3.8}$$

4. SELF-SIMILAR ASYMPTOTICS

In the following we assume that the kernel $B(\cos\theta)$ and the dimension $d \geqslant 2$ are fixed. Let the initial characteristic function be isotropic, $\phi_0 = \phi_0(|\mathbf{k}|)$. Then the asymptotic expansion (3.8) reduces to

$$\phi(|\mathbf{k}|, t) = 1 - \frac{|\mathbf{k}e^{-\mu t}|^2}{2} + O(|\mathbf{k}|^{2(1+\delta)}), \qquad \mu = \frac{\lambda_0}{2}, \quad |\mathbf{k}| \to 0, \tag{4.1}$$

and (when the higher order terms are omitted) can be considered as a function of the self-similar variable

$$x = |\mathbf{k}| e^{-\mu t}. \tag{4.2}$$

This remark leads to the problem of the existence of a self-similar solution

$$\phi(|\mathbf{k}|, t) = \psi(x) = \psi(|\mathbf{k}| e^{-\mu t}) \tag{4.3}$$

of Eq. (2.7). This question was discussed in detail in a previous paper,[8] where the following statement was proved (we slightly re-formulate the result of ref. 8, Theorem 7.1, in a form convenient to our goals):

Proposition 4.1.

(A) There exists a unique self-similar solution (4.3) of Eq. (2.7) satisfying two assumptions: $\psi(|\mathbf{k}|)$ is bounded for $|\mathbf{k}| \in \mathbb{R}_+$ and

$$\psi(|\mathbf{k}|) = 1 - \tfrac{1}{2} |\mathbf{k}|^2 + \cdots, \qquad |\mathbf{k}| \to 0, \tag{4.4}$$

(B) The function $\psi(|\mathbf{k}|)$, defined in (A), is a characteristic function such that

$$e^{-|\mathbf{k}|^2/2} \leqslant \psi(|\mathbf{k}|) \leqslant e^{-|\mathbf{k}|}(1 + |\mathbf{k}|), \qquad \mathbf{k} \in \mathbb{R}^d. \tag{4.5}$$

The corresponding solution of the kinetic equation (2.1) reads

$$f(|\mathbf{v}|, t) = e^{d\mu t} F(|\mathbf{v}| e^{\mu t}), \qquad F(|\mathbf{v}|) = (2\pi)^{-d} (\psi, e^{i\mathbf{k}\cdot\mathbf{v}}), \qquad \mathbf{v} \in \mathbb{R}^d. \tag{4.6}$$

(C) The function $F(|\mathbf{v}|)$ has a power-like decay, as $|\mathbf{v}| \to \infty$, for "almost all" kernels $B(\cos\theta)$ in Eq. (2.2).

Remark. Proposition 4.3 is formulated for a given kernel $B(\cos\theta)$ satisfying Eq. (2.3) and for a given dimension $d \geqslant 2$. The functions $\psi(|\mathbf{k}|)$ and $F(|\mathbf{v}|)$ are obviously different for different choices of B and d.

Then, having uniquely defined the relevant self-similar solution (4.3) by Proposition 4.1, we can formulate our goal in clear terms: we must prove that

$$\phi(\mathbf{k}e^{\mu t}, t) \xrightarrow[t \to \infty]{} \psi(|\mathbf{k}|), \qquad \mathbf{k} \in \mathbb{R}^d. \tag{4.7}$$

(note that the pointwise convergence is sufficient for our goals since $\phi(\mathbf{k}e^{\mu t}, t)$ is, for any fixed $t > 0$, a characteristic function.) We expect that the limit (4.7) can be proved under the standard assumptions (2.3), (2.8) (the latter guarantee the leading terms of the asymptotics (3.8)) and the only essential restriction (3.8).

The idea of the proof is the same as in ref. 8. If Eq. (4.7) is true for two solutions $\phi_{1,2}(\mathbf{k}, t)$ satisfying (2.8) (3.8) then

$$\lim_{t \to \infty} |\phi_1(\mathbf{k}e^{\mu t}, t) - \phi_2(\mathbf{k}e^{\mu t}, t)| = 0, \qquad \mathbf{k} \in \mathbb{R}^d. \tag{4.8}$$

Therefore, if we denote:

$$u = \phi_1 - \phi_2, \qquad \Phi = \tfrac{1}{2}(\phi_1 + \phi_2) \tag{4.9}$$

and consider the equation for $u(x, t)$. By the elementary identity

$$ab - cd = \tfrac{1}{2}[(a+c)(b-d) + (b+d)(a-c)] \tag{4.10}$$

and Eq. (2.7) (in the notation (3.3)) we obtain

$$\frac{\partial u}{\partial t} + u = \langle \Phi(\mathbf{k}_+)\, u(\mathbf{k} - \mathbf{k}_+) + \Phi(\mathbf{k} - \mathbf{k}_+)\, u(\mathbf{k}_+) \rangle,$$
$$u_{|t=0} = u_0(\mathbf{k}) = \phi_1(\mathbf{k}, 0) - \phi_2(\mathbf{k}, 0) \tag{4.11}$$

The obvious estimate

$$\Phi \leqslant \tfrac{1}{2}(|\phi_1| + |\phi_2|) \leqslant 1 \tag{4.12}$$

leads to the inequality

$$\left|\frac{\partial u}{\partial t}+u\right| \leqslant \hat{L}^{+}|u|, \qquad \hat{L}^{+}=\hat{L}+\hat{I}, \qquad u_{|t=0}=u_0 \tag{4.13}$$

where $\hat{I}$ is the identity, $\hat{L}$ is the linearized operator defined in Eq. (3.2). If $u=ye^{-t}$ then

$$\left|\frac{\partial y}{\partial t}\right| \leqslant \hat{L}^{+}|y|, \qquad y_{|t=0}=u_0(\mathbf{k}) \tag{4.14}$$

On the other hand for any complex function $y(\mathbf{k}, t)$ ($\mathbf{k} \in \mathbb{R}^d$ is fixed)

$$|y|\left|\frac{\partial |y|}{\partial t}\right| = \frac{1}{2}\left|\frac{\partial}{\partial t}(yy^*)\right| = \frac{1}{2}\left|y\frac{\partial y^*}{\partial t}+y^*\frac{\partial y}{\partial t}\right| \leqslant |y|\left|\frac{\partial y}{\partial t}\right|,$$

where y^* is the complex conjugate of y. Therefore (note that y and its time derivative are continuous functions of $\mathbf{k} \in \mathbb{R}^d$ and $t \in \mathbb{R}_+$):

$$\left|\frac{\partial |y|}{\partial t}\right| \leqslant \left|\frac{\partial y}{\partial t}\right| \leqslant \hat{L}^{+}|y|, \qquad y_{|t=0}=u_0(\mathbf{k}). \tag{4.15}$$

The operator $\hat{L}^{+}$ is positive ($\hat{L}^{+}y \geqslant 0$ for any $y \geqslant 0$). Hence

$$|y(\mathbf{k}, t)| \leqslant e^{t\hat{L}^{+}}|u_0(\mathbf{k})|, \tag{4.16}$$

or, equivalently,

$$|u(\mathbf{k}, t)| \leqslant e^{t\hat{L}}|u_0(\mathbf{k})|, \tag{4.17}$$

in the notation (3.2). This simple consideration already leads to the proof of Eq. (4.7) under the additional restriction

$$q_{\alpha\beta}(0)=\left(f_0, v_\alpha v_\beta-\frac{|\mathbf{v}|^2}{d}\delta_{\alpha\beta}\right)=0. \tag{4.18}$$

In such a case

$$u_0(\mathbf{k})=O(|\mathbf{k}|^{2(1+\delta)}), \quad |\mathbf{k}| \to 0; \qquad |u_0(\mathbf{k})| \leqslant |\phi_1(\mathbf{k}, 0)|+|\phi_2(\mathbf{k}, 0)| \leqslant 2. \tag{4.19}$$

Hence for any $0<\epsilon<\delta$ there exists a constant A_ϵ such that

$$|u_0(\mathbf{k})| \leqslant A_\epsilon |\mathbf{k}|^{2(1+\epsilon)}, \tag{4.20}$$

and therefore

$$(e^{t\hat{L}}\,|u_0|)(\mathbf{k}) \leqslant A_\epsilon\,|\mathbf{k}|^{2(1+\epsilon)}\,e^{-\lambda(1+\epsilon)\,t}, \tag{4.21}$$

where

$$\lambda(p) = \int_{S^{d-1}} d\mathbf{n}\,B(\hat{\mathbf{k}}\cdot\mathbf{n})\{1 - z^{2p}(\hat{\mathbf{k}}\cdot\mathbf{n})^{2p} - [1-\beta(\hat{\mathbf{k}}\cdot\mathbf{n})^2]^p\},$$

$$\hat{\mathbf{k}} \in S^{d-1}, \quad \beta = z(2-z), \quad p \geqslant 1, \tag{4.22}$$

($\lambda(p)$ can be expressed through a 1-d integral, but we do not need it). Thus we obtain from the estimate (4.17)

$$|u(\mathbf{k}, t)| \leqslant A_\epsilon\,|\mathbf{k}|^{2(1+\epsilon)}\,e^{-\lambda(1+\epsilon)\,t}, \tag{4.23}$$

for any $\mathbf{k} \in \mathbb{R}^d$, $t \in \mathbb{R}$) $0 < \epsilon < \delta$ provided the condition (4.18) is fulfilled.

We note that

$$\mu = \frac{\lambda}{2} = \frac{1}{2}\,\lambda(1) \tag{4.24}$$

in the notation (3.5), (4.1), (4.22). Hence

$$|u(\mathbf{k}e^{\mu t}, t)| \leqslant A_\epsilon\,|\mathbf{k}|^{2(1+\epsilon)}\,e^{-\Delta(\epsilon)\,t}, \qquad \Delta(\epsilon) = \lambda(1+\epsilon) - (1+\epsilon)\,\lambda(1). \tag{4.25}$$

The inequality $\Delta(\epsilon) > 0$ was already proved (in a more general case) in ref. 8 (Lemma 6.1). For the reader's convenience we also present a simplified proof here. It is sufficient to show that

$$\Delta'(0) = \lambda'(1) - \lambda(1) > 0. \tag{4.26}$$

A simple calculation leads to the equality

$$\Delta'(0) = \int_{S^{d-1}} d\mathbf{n}\,B(\hat{\mathbf{k}}\cdot\mathbf{n})\,F[(\hat{\mathbf{k}}\cdot\mathbf{n})^2, z], \tag{4.27}$$

where

$$F(x; z) = a(z^2 x) + a(1-\beta x) - 1,$$

$$a(x) = x(1 - \log x), \qquad 0 \leqslant x \leqslant 1, \quad \tfrac{1}{2} \leqslant z \leqslant 1,$$

$$\beta = z(2-z) = 1 - (1-z)^2.$$

We consider $F(x; z)$ as a function of $x \in [0, 1]$ for a fixed $1/2 \leqslant z \leqslant 1$. It is easy to verify that

$$F(0; z) = 0, \qquad F(1; z) = a(z^2) + a[(1-z)^2)] - 1;$$

$$F_x(x; z) \xrightarrow[x \to 0]{} \infty, \qquad F_{xx}(x; z) < 0,$$

where the index x denotes a partial derivative with respect to x. Hence, $F(x; z)$ cannot have local minima in (0, 1). Therefore $F(x; z) > 0$ for all $x \in (0, 1)$ provided $F(1; z) \geqslant 0$. Noting that

$$F(1; 1) = 0, \qquad F_z(1; z) = 4[h(1-z) - h(z)], \qquad h(z) = z \log z,$$

we observe that $h'(z) > 0$ if $z > e^{-1}$. Hence $F_z(1; z) < 0$ for $z \in (1/2, 1]$ and $F_z(1; 1/2) = 0$. Therefore

$$F(1; z) > 0; \qquad \Rightarrow F(x; z) > 0, \;\; \tfrac{1}{2} \leqslant z < 1 \quad 0 < x \leqslant 1.$$

Hence $\Delta' > 0$, (see Eq. (4.27)) provided $B(\cos\theta)$ is not concentrated at $\theta = \pi/2$. The latter case, however, is not interesting at all since it leads to the equality $\lambda_0 = 2\mu = 0$ (see Eq. (4.24)). To exclude such case it is enough to assume $B(x) \in L^1_+([-1, 1])$.

Hence the inequality $\Delta(\epsilon) > 0$ holds in all cases when $\mu > 0$. Then we conclude, by the estimate (4.25) and the definition (4.24) that

$$\lim_{t \to \infty} |u(\mathbf{k}e^{\mu t}, t)| = 0, \qquad \mathbf{k} \in \mathbb{R}^d, \tag{4.28}$$

where $u(\mathbf{k}, t)$ denotes the difference between any two solutions of the problem (2.7) with an initial distribution function satisfying the condition (4.18). Taking the self-similar solution $\psi(\mathbf{k}e^{\mu t})$ as one of the two we obtain the following result

Lemma 4.2. Let $f(\mathbf{v}, t)$ be any solution of the problem (2.1) satisfying the conditions (2.8) and (4.18). Then its characteristic function $\phi(\mathbf{k}, t) = (f, e^{-i\mathbf{k}\cdot\mathbf{v}})$ has the self-similar asymptotics

$$\lim_{t \to \infty} \phi(\mathbf{k}e^{\mu t}, t) = \psi(\mathbf{k}), \qquad \mathbf{k} \in \mathbb{R}^d,$$

where $\psi(\mathbf{k})$ is defined in Proposition 4.3.

In the next section we shall see that the extra condition (4.18) can be removed.

5. COMPLETE PROOF OF THE CONJECTURE

Let us consider the most general case. We return to the (exact!) Eq. (4.11) and represent $u(\mathbf{k}, t)$ as a sum

$$u(\mathbf{k}, t) = u_1(\mathbf{k}, t) + w(\mathbf{k}, t)$$
$$u_1 = \frac{1}{2} b_{\alpha\beta}(t) \left(k_\alpha k_\beta - \frac{|\mathbf{k}|^2}{d} \delta_{\alpha\beta} \right), \quad b_{\alpha\beta} = q^1_{\alpha\beta} - q^2_{\alpha\beta}, \quad w = O(|\mathbf{k}|^{2(1+\delta)}) \tag{5.1}$$

We also note that, in the notations (3.5), (4.13),

$$b_{\alpha\beta}(t) = b_{\alpha\beta}(0)\, e^{-\lambda_1 t}, \qquad \frac{\partial u_1}{\partial t} + u_1 = \hat{L}^+ u_1, \tag{5.2}$$

Then we obtain from Eq. (4.11)

$$\frac{\partial w}{\partial t} + w = \langle \Phi(\mathbf{k}_+)\, w(\mathbf{k} - \mathbf{k}_+) + \Phi(\mathbf{k} - \mathbf{k}_+)\, w(\mathbf{k}_+) \rangle$$
$$\langle [1 - \Phi(\mathbf{k}_+)]\, u_1(\mathbf{k} - \mathbf{k}_+) + [1 - \Phi(\mathbf{k} - \mathbf{k}_+)]\, u_1(\mathbf{k}_+) \rangle, \quad w_{|t=0} = w_0(\mathbf{k}) \tag{5.3}$$

In order to estimate the source term we note that

$$|\Phi(\mathbf{k})| \leqslant 1, \quad |1 - \Phi(\mathbf{k})| \leqslant 2; \qquad 1 - \Phi(\mathbf{k}) = O(|\mathbf{k}|^2), \quad \mathbf{k} \to 0$$

Hence for any $0 < \epsilon < 2$ there exists a constant B_ϵ such that

$$|1 - \Phi(\mathbf{k})| \leqslant B_\epsilon\, |\mathbf{k}|^{2\epsilon}, \qquad \mathbf{k} \in \mathbb{R}^d.$$

On the other hand,

$$|u_1(\mathbf{k}, t)| \leqslant \text{const.}\, |\mathbf{k}|^2\, e^{-\lambda_1 t}, \qquad \mathbf{k} \in \mathbb{R}^d.$$

Therefore

$$\left| \frac{\partial w}{\partial t} + w \right| \leqslant \hat{L}^+\, |w| + C_\epsilon\, |\mathbf{k}|^{2(1+\epsilon)}\, e^{-\lambda_1 t}$$

The same arguments as in Section 4 (see derivation of Eq. (4.17) from Eq. (4.13)) lead to the estimate

$$\frac{\partial |w|}{\partial t} \leqslant \hat{L}\, |w| + C_\epsilon\, |\mathbf{k}|^{2(1+\epsilon)}\, e^{-\lambda_1 t}$$

Hence (note that $e^{t\hat{L}}$ is a positive operator) we obtain

$$|w(\mathbf{k}, t)| \leqslant e^{t\hat{L}}\, |w_0(\mathbf{k})| + C_\epsilon \int_0^t d\tau\, e^{-\lambda_1(t-\tau)}\, e^{\tau\hat{L}}\, |\mathbf{k}|^{2(1+\epsilon)} \tag{5.4}$$

The first term was already estimated in Eq. (4.21) (we assume that the condition (4.20) holds for $w_0(\mathbf{k})$). The second term, which we denote by $I_\epsilon(\mathbf{k}, t)$, reads

$$\begin{aligned} I_\epsilon(\mathbf{k}, t) &= C_\epsilon \int_0^t d\tau\, e^{-\lambda_1(t-\tau)} e^{-\lambda(1+\epsilon)\,\tau}\, |\mathbf{k}|^{2(1+\epsilon)} \\ &= C_\epsilon\, \frac{|\mathbf{k}|^{2(1+\epsilon)}}{\lambda_1 - \lambda(1+\epsilon)}\, [e^{-\lambda(1+\epsilon)\,t} - e^{-\lambda_1 t}]. \end{aligned}$$

We note that $\lambda(1+\epsilon) \to \lambda(1) = \lambda_0$, as $\epsilon \to 0$, whereas $\lambda_1 > \lambda_0$ (see Eq. (3.5)) does not depend on ϵ. Therefore we obtain (for sufficiently small $\epsilon > 0$) the same upper estimate

$$|I_\epsilon(\mathbf{k}, t)| \leqslant D_\epsilon\, |\mathbf{k}|^{2(1+\epsilon)}\, e^{-\lambda(1+\epsilon)\,t},$$

as we got already in (4.21) for the first term in the right hand side of (5.4). Hence

$$|w(\mathbf{k}, t)| \leqslant E_\epsilon\, |\mathbf{k}|^{2(1+\epsilon)}\, e^{-\lambda(1+\epsilon)\,t},$$

and (see Eqs. (5.1) and (5.2))

$$|u(\mathbf{k}, t)| = |u_1(\mathbf{k}, t) + w(\mathbf{k}, t)| \leqslant E_\epsilon\, |\mathbf{k}|^{2(1+\epsilon)}\, e^{-\lambda(1+\epsilon)\,t} + C_\epsilon\, |\mathbf{k}|^2\, e^{-\lambda_1 t}.$$

Therefore

$$|u(\mathbf{k}e^{\mu t}, t)| \leqslant E_\epsilon\, |\mathbf{k}|^{2(1+\epsilon)}\, e^{-\Delta(\epsilon)\,t} + C_\epsilon\, |\mathbf{k}|^2\, e^{-(\lambda_1 - \lambda_0)\,t},$$

in the notation (4.25). The inequality $\Delta(\epsilon) > 0$ was proved in Section 4, and we already mentioned that $\lambda_1 > \lambda_0 = 2\mu$. Thus the equality

$$\lim_{t \to \infty} |\phi_1(\mathbf{k}e^{\mu t}, t) - \phi_2(\mathbf{k}e^{\mu t}, t)| = 0, \qquad \mathbf{k} \in \mathbb{R}^d \tag{5.5}$$

is proved for any two solutions of the problem (2.7) without the assumption $q^1_{\alpha\beta} = q^2_{\alpha\beta} = 0$ used in Section 5. If we replace $\phi_2(\mathbf{k}, t)$ by the self-similar solution $\psi(\mathbf{k}e^{-\mu t})$ and $\phi_1(\mathbf{k}, t)$ by $\phi(\mathbf{k}, t)$, then our result reads

$$\lim_{t \to \infty} \phi(\mathbf{k}e^{\mu t}, t) = \psi(\mathbf{k}), \qquad \mathbf{k} \in \mathbb{R}^d$$

where $\phi_0(\mathbf{k}) = \phi(\mathbf{k}, 0)$ satisfies the most general assumptions of Propositions 2.1 and 2.2.

We already mentioned in Section 4 that the pointwise convergence of the characteristic functions implies the weak convergence of distribution functions. Therefore we can formulate the final result in terms of solutions of the kinetic equation (2.1):

Theorem 5.1. Let $f(\mathbf{v}, t)$ be the solution of the problem (2.1), where

$$(f_0, 1) = 1, \quad (f_0, \mathbf{v}) = 0, \quad (f_0, |\mathbf{v}|^2) = d, \quad (f_0, |\mathbf{v}|^{2+\delta}) < \infty, \qquad \delta > 0.$$

Then

$$e^{-d\mu t} f(\mathbf{v}e^{-\mu t}, t) \to F(\mathbf{v}) = 0, \qquad t \to \infty, \quad d \geqslant 2,$$

where the convergence is understood in the sense of probability measures and the existence of the limiting measure, $F(\mathbf{v})$ and μ are defined in Proposition 4.3.

Proof. It is already given above. We just note that the term "solution" is understood in the same sense as in Proposition 2.1.

Thus, the Ernst–Brito conjecture is proved for practically all initial conditions having interest in the applications.

ACKNOWLEDGMENTS

The research described in the paper was supported by MURST of Italy and NFR of Sweden.

REFERENCES

1. A. V. Bobylev and C. Cercignani, Exact eternal solutions of the Boltzmann equation, *J. Stat. Phys.* **106**:1019–1038 (2002).
2. A. V. Bobylev and C. Cercignani, Self-similar solutions of the Boltzmann equation and their applications, *J. Stat. Phys.* **106**:1039–1071 (2002).
3. M. H. Ernst and R. Brito, High energy tails for inelastic Maxwell models, preprint (submitted to *Europhys. Lett.*, 2001).
4. M. H. Ernst and R. Brito, Scaling solutions of inelastic Boltzmann equations with overpopulated high energy tails, preprint (submitted to *J. Stat. Phys.*, 2002).
5. A. Baldassarri, U. Marini Bettolo Marconi, and A. Puglisi, Influence of correlations on the velocity statistics of scalar granular gases, *Europhys. Lett.* **58**:14–20 (2002).
6. P. L. Krapivski and E. Ben-Naim, Nontrivial velocity distributions in inelastic gases, *J. Phys. A: Math. Gen.* **35**:L147–L152 (2002).

7. A. V. Bobylev, J. A. Carrillo, and I. A. Gamba, On some properties of kinetic and hydrodynamic equations for inelastic interaction, *J. Stat. Phys.* **98**:743–773 (2000).
8. A. V. Bobylev and C. Cercignani, Self-similar asymptotics for the Boltzmann equation with inelastic and elastic interactions, *J. Stat. Phys.* **110**:333–375 (2003).
9. E. Gabetta, G. Toscani, and B. Wennberg, Metrics for probability distributions and the trend to equilibrium for solutions of the Boltzmann equation, *J. Stat. Phys.* **81**:901–934 (1995).
10. A. V. Bobylev, The Fourier transform methods in the theory of the Boltzmann equation for Maxwell molecules, *Dokl. Akad. Nauk SSSR* **225**:1041–1044 (1975) [In Russian].
11. A. V. Bobylev, The theory of the nonlinear spatially uniform Boltzmann equation for Maxwell molecules, *Sov. Sci Rev.* **C7**:111–233 (1988).
12. W. Feller, *An Introduction to Probability Theory and its Applications*, Vol. 1 (Wiley, N.Y., 1966).
13. A. V. Bobylev and C. Cercignani, Moment equations for a granular material in a thermal bath, *J. Stat. Phys.* **106**:547–567 (2002).